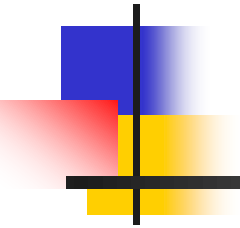
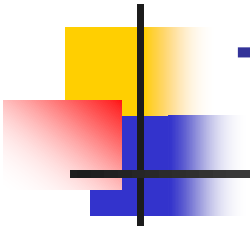
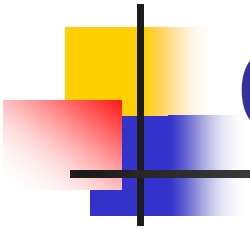


Algorithms and Data Structures
Lecture XIV



This Lecture

- Introduction to computational geometry
  - Algorithms
    - Points and lines in 2D space
    - *Line sweep* technique
    - Closest pair of points using divide-and-conquer
  - A peek at data structures for mutidimensional range searching



Computational Geometry

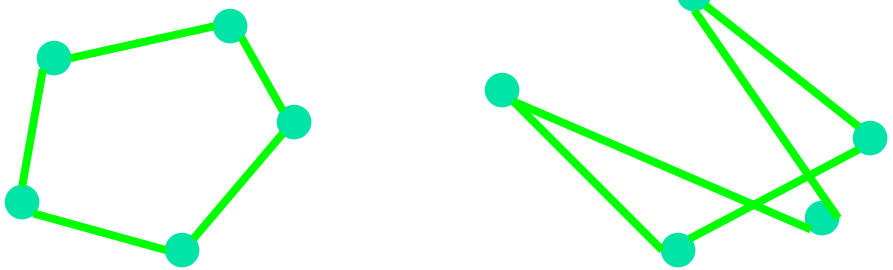
- The term first appeared in the 70's
- Originally referred to computational aspects of solid/geometric modeling
- Later as the field of algorithm design and analysis of discrete geometry
- Algorithmic bases for many scientific & engineering disciplines
  - GIS, robotics, computer graphics, computer vision, CAD/CAM, VLSI design, etc.

Geometric Objects in Plane

- **Point**: defined by a pair of coordinates (x,y)
- **Segment**: portion of a straight line between two points – their *convex combination*

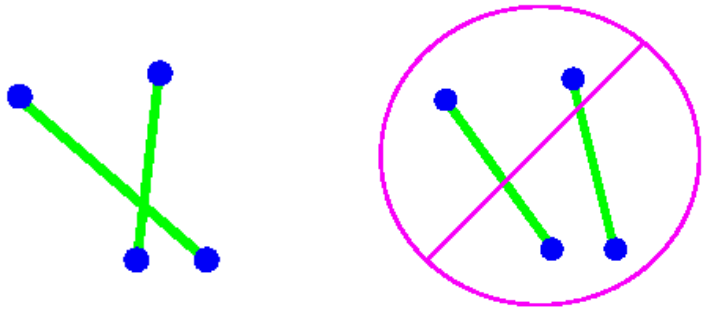
$$\begin{cases} x = \alpha x_A + (1-\alpha) x_B \\ y = \alpha y_A + (1-\alpha) y_B \end{cases}$$

- **Polygon**: a circular sequence of points (vertices) and segments (edges) between them

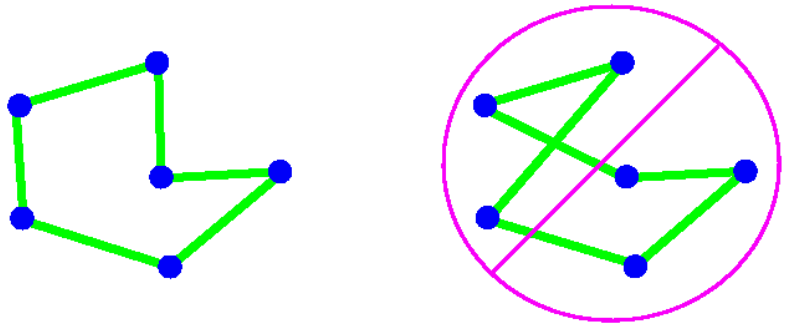


Some Geomtric Problems

- **Segment intersection**: Given two segments, do they intersect?

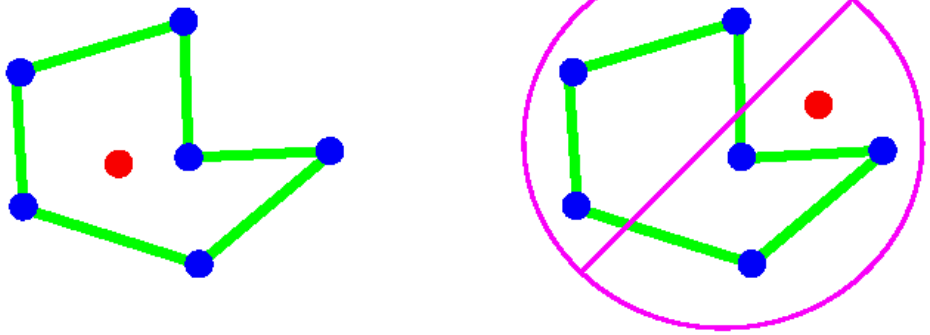


- **Simple closed path**: Given a set of points, find a non-intersecting polygon with vertices on the points



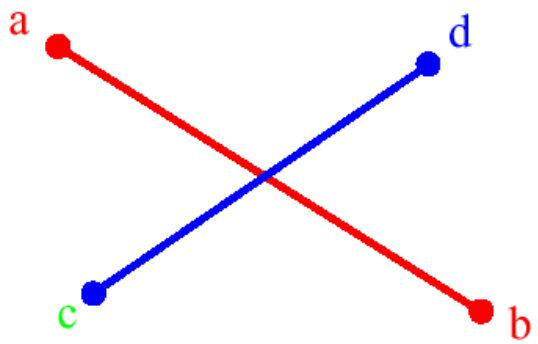
Some Geomtric Problems (2)

- **Inclusion in polygon**:Is a point inside or outside a polygon?



Segment Intersection

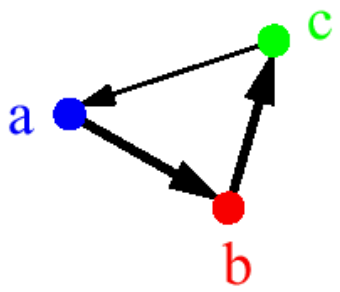
- Test whether segments (a,b) and (c,d) intersect. ***How do we do it?***



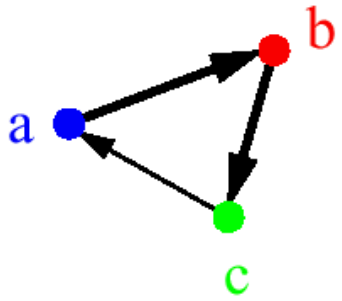
- We could start by writing down the **equations of the lines through the segments**, then test whether the lines intersect, then ...
- An alternative (and simpler) approach is based in the notion of **orientation of an ordered triplet of points** in the plane

Orientation in the Plane

- The orientation of an ordered triplet of points in the plane can be
  - counterclockwise (left turn)
  - clockwise (right turn)
  - collinear (no turn)



counterclockwise
(left turn)

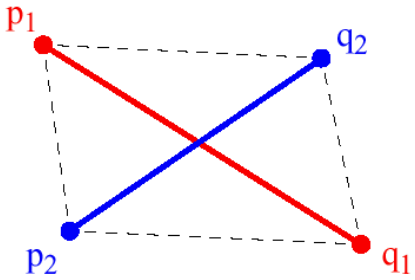


clockwise
(right turn)

collinear
(no turn)

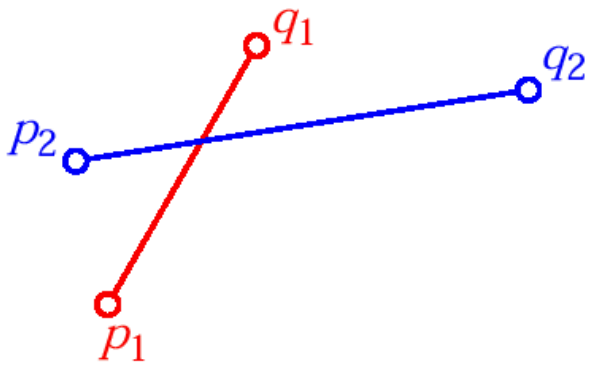
Intersection and Orientation

- Two segments (p_1,q_1) and (p_2,q_2) intersect if and only if one of the following two conditions is verified
- General case:
  - (p_1,q_1,p_2) and (p_1,q_1,q_2) have different orientations **and**
  - (p_2,q_2,p_1) and (p_2,q_2,q_1) have different orientations
- Special case
  - (p_1,q_1,p_2), (p_1,q_1,q_2), (p_2,q_2,p_1), and (p_2,q_2,q_1) are all collinear **and**
  - the x-projections of (p_1,q_1) and (p_2,q_2) intersect
  - the y-projections of (p_1,q_1) and (p_2,q_2) intersect

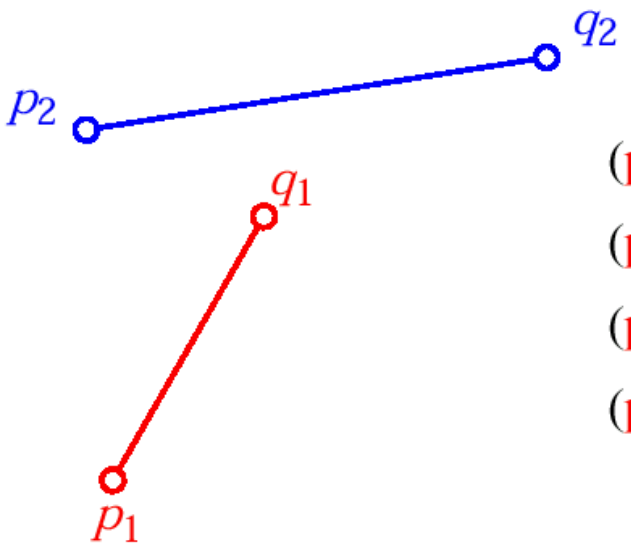


Orientation Examples

- General case:
  - (p_1,q_1,p_2) and (p_1,q_1,q_2) have different orientations **and**
  - (p_2,q_2,p_1) and (p_2,q_2,q_1) have different orientations

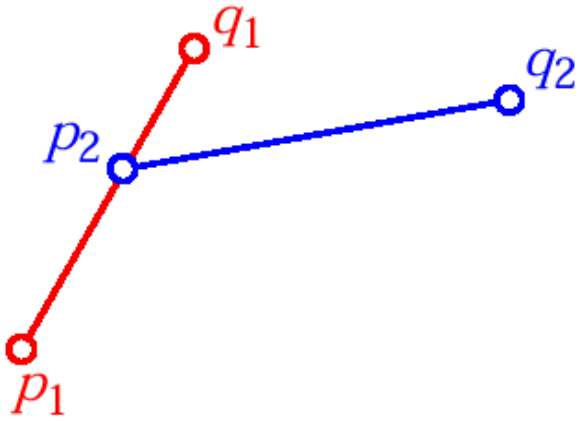


(p_1,q_1,p_2)
(p_1,q_1,q_2)
(p_2,q_2,p_1)
(p_2,q_2,q_1)



(p_1,q_1,p_2)
(p_1,q_1,q_2)
(p_2,q_2,p_1)
(p_2,q_2,q_1)

Orientation Examples (2)

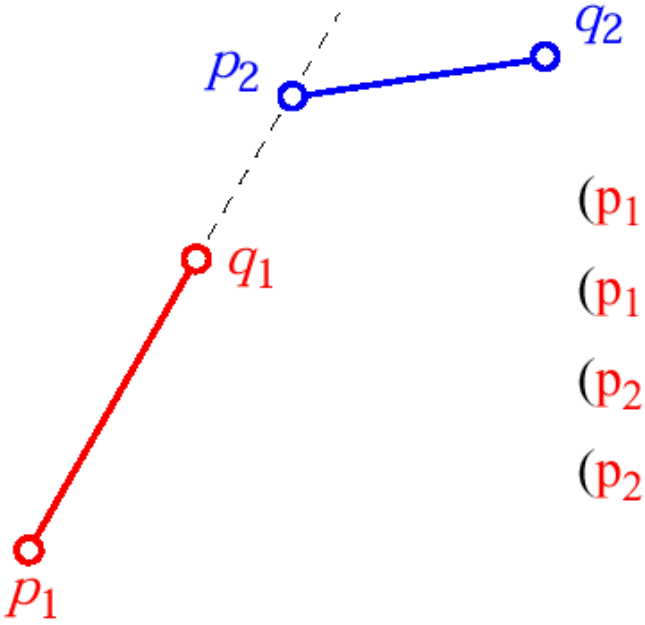


(p_1,q_1,p_2)

(p_1,q_1,q_2)

(p_2,q_2,p_1)

(p_2,q_2,q_1)



(p_1,q_1,p_2)

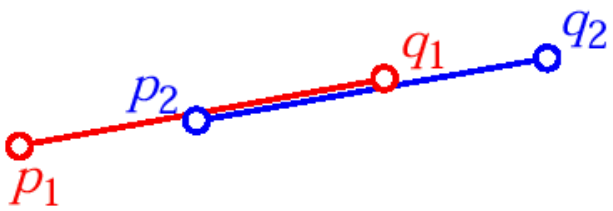
(p_1,q_1,q_2)

(p_2,q_2,p_1)

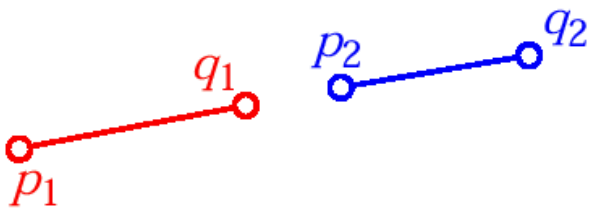
(p_2,q_2,q_1)

Orientation Examples (3)

- Special case
  - (p_1,q_1,p_2), (p_1,q_1,q_2), (p_2,q_2,p_1), and (p_2,q_2,q_1) are all collinear **and**
  - the x-projections of (p_1,q_1) and (p_2,q_2) intersect
  - the y-projections of (p_1,q_1) and (p_2,q_2) intersect



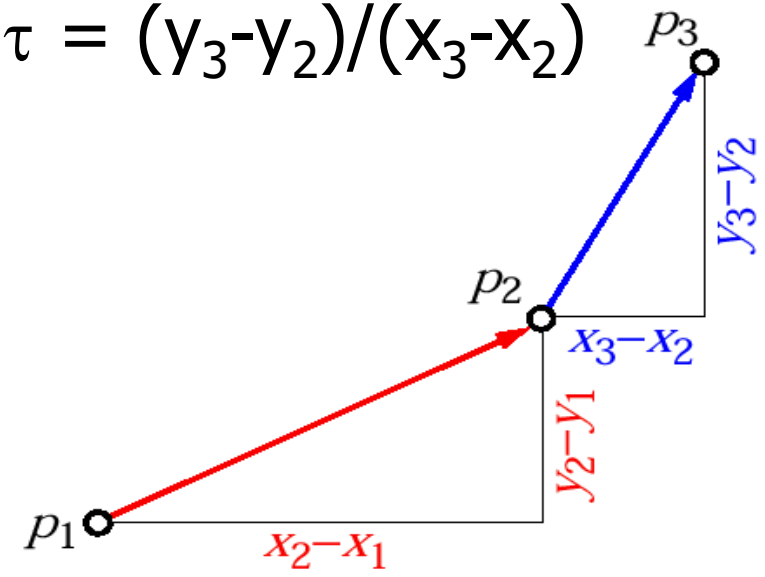
(p_1,q_1,p_2)
(p_1,q_1,q_2)
(p_2,q_2,p_1)
(p_2,q_2,q_1)



(p_1,q_1,p_2)
(p_1,q_1,q_2)
(p_2,q_2,p_1)
(p_2,q_2,q_1)

Computing the Orientation

- slope of segment (p_1, p_2): $\sigma = (y_2-y_1)/(x_2-x_1)$
- slope of segment (p_2, p_3): $\tau = (y_3-y_2)/(x_3-x_2)$



- Orientation test
  - counterclockwise (left turn): $\sigma < \tau$
  - clockwise (right turn): $\sigma > \tau$
  - collinear (no turn): $\sigma = \tau$

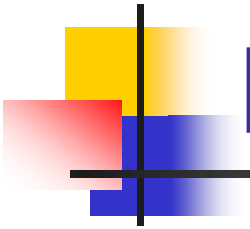
Computing the Orientation (2)

- The orientation depends on whether the following expression is positive, negative, or null

  $(y_2-y_1)(x_3-x_2) - (y_3-y_2)(x_2-x_1) = ?$

- This is a *cross product* of two vectors

$$(x_2 - x_1, y_2 - y_1) \times (x_3 - x_2, y_3 - y_2) = \det \begin{pmatrix} x_2 - x_1 & x_3 - x_2 \\ y_2 - y_1 & y_3 - y_2 \end{pmatrix}$$

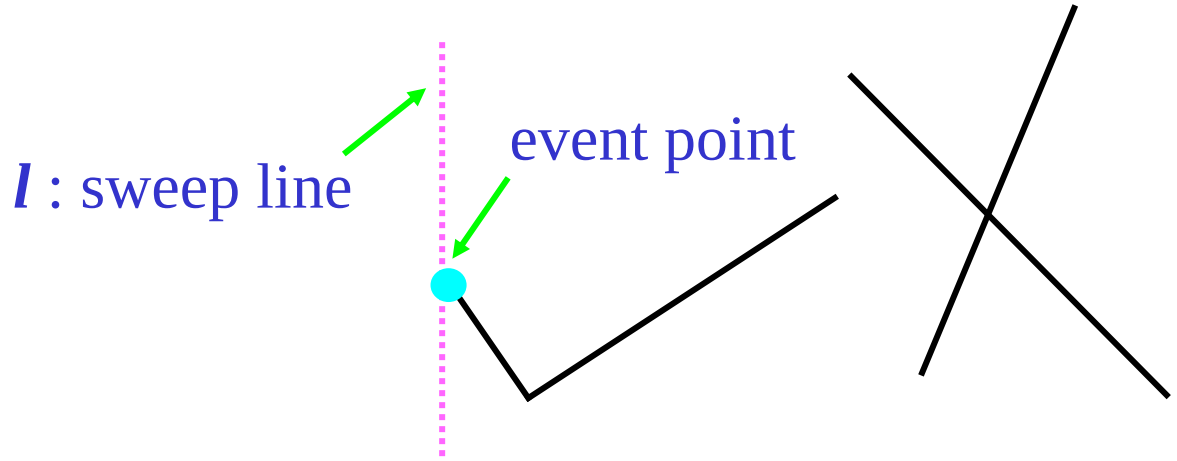


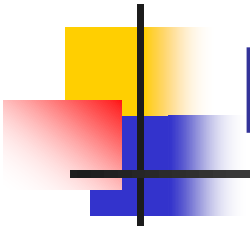
Determining Intersections

- Given a set of *n* segments, we want to determine whether any two line segments intersect
- A **brute force approach** would take $O(n^2)$ time (testing each with every other segment for intersection)

Determining Intersections (2)

- It can be done faster by using a powerfull comp. geometry technique, called ***plane-sweeping***. Two sets of data are maintained:
  - *sweep-line status*: the set of segments intersecting the sweep line *l*
  - *event-point schedule*: where updates to *l* are required

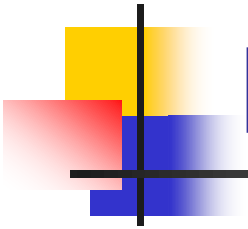




Plane Sweeping Algorithm

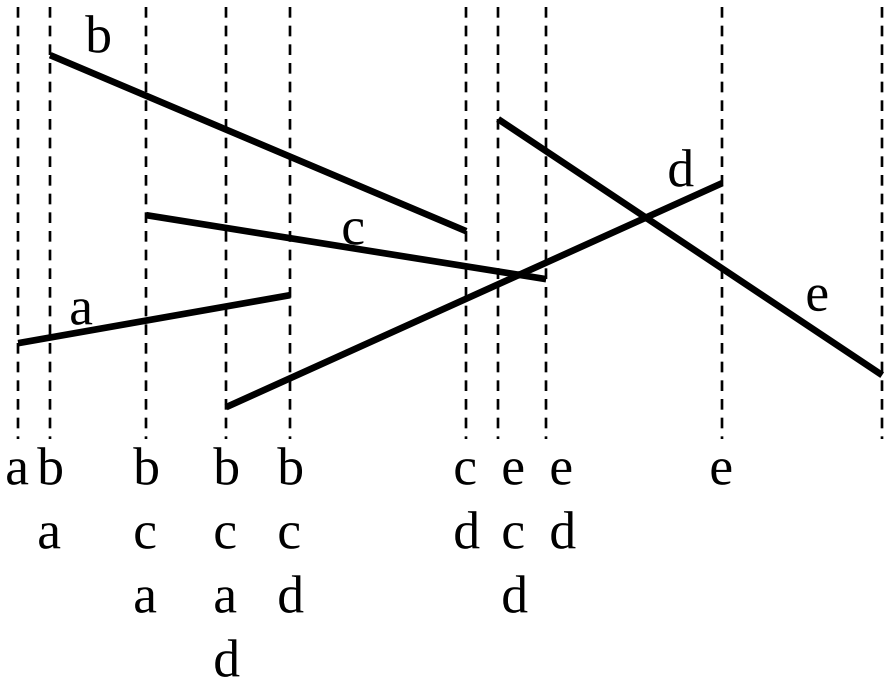
- Each segment end point is an event point
- At an event point, update the status of the sweep line & perform intersection tests
  - left end point: a new segment is added to the status of / and it's tested against the rest
  - right end point: it's deleted from the status of /
- Only testing pairs of segments for which there is a horizontal line that intersects both segments
- This might be not good enough. It may still be inefficient, $O(n^2)$ for some cases

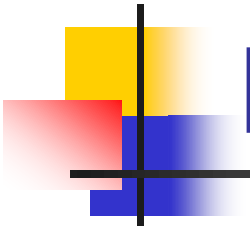
Plane Sweep Algorithm (2)



- To include the idea of being close in the horizontal direction, only test segments that are **adjacent in the vertical direction**
- Only test a segment with the **ones above and below** (predecessor and successor, *rings a bell*…)
  - the ”above-below” relationship among segments does not change unless there is an intersection
- New “status”: *ordered* sequence of segments

Plane Sweeping Algorithm (3)





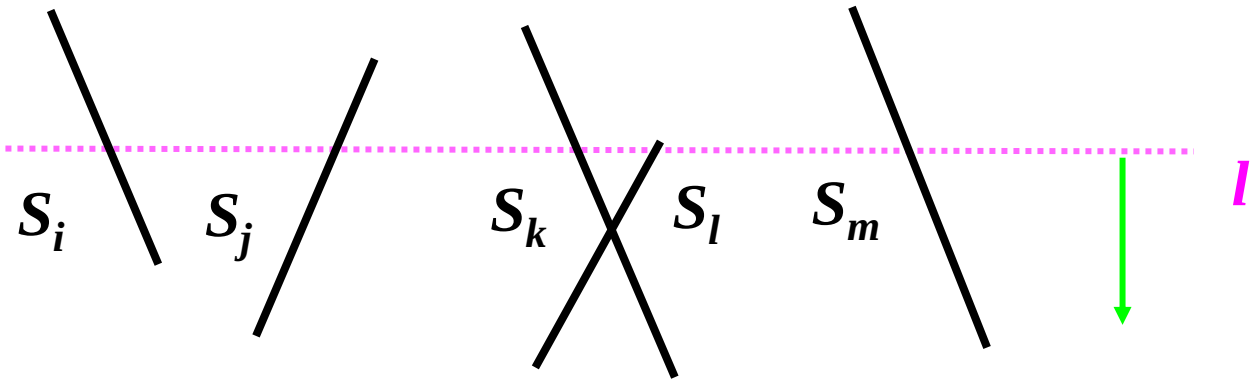
Pseudo Code

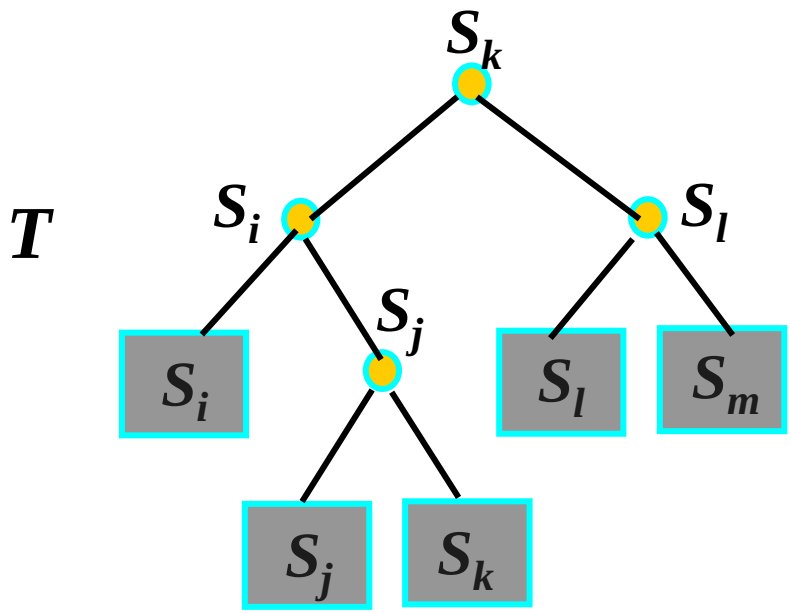
```
AnySegmentsIntersect(S)
01 T ← ∅
02 sort the end points of the segments in S from left
   to right, breaking ties by putting points with lower
   y-coords first
03 for each point p in the sorted list of end points do
04    if p is the left end point of a segment s then
05       Insert(T,s)
06          if (Above(T,s) exists and intersects s) or
               (Below(T,s) exists and intersects s) then
07             return TRUE
08    if p is the right end point of a segment s then
09       if both Above(T,s) and Below(T,s) exist and
            Above(T,s) intersects Below(T,s) then
10             return TRUE
11          Delete(T,s)
12 return FALSE
```

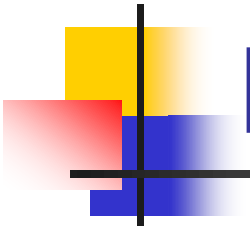
Implementing Event Queue

- Define an order on event points
- Store the event points in a priority queue
  - both insertion or deletion takes O(log *n*) time, where *n* is the number of events, and fetching minimum – O(1)
  - For our algorithm sorted array is enough, because the set of events does not change.
- Maintain the status of l using a binary search tree T
  - the up-to-down order of segments on the line l <=> the left-to-right order of leaves in T
  - segments in internal nodes guide search
  - each update and search takes O(log *n*)

Status and Structure





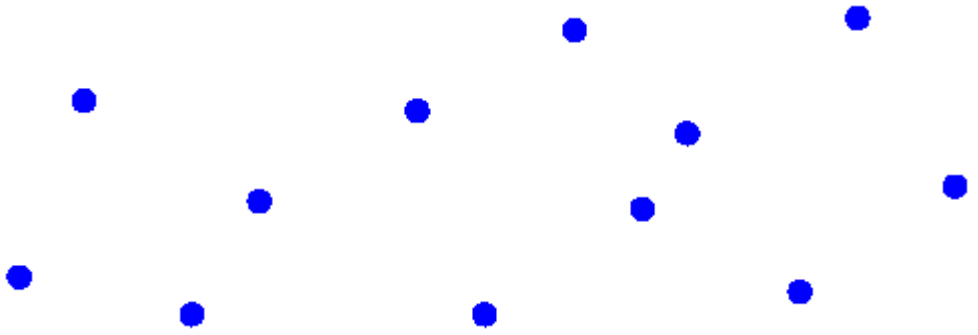


Running Time

- Sorting the segments takes O(n log n) time
- The loop is executed once for every end point ($2n$) taking each time O(log n) time (e.g., red-black tree operation)
- The total running time is O(n log n)

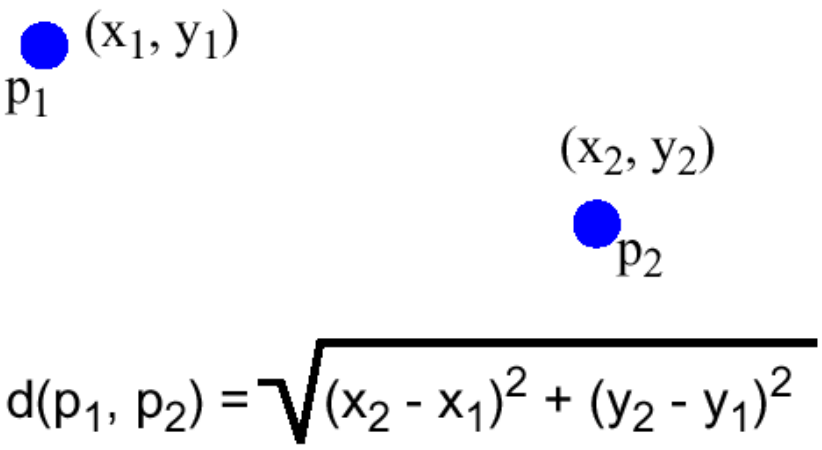
Closest Pair

- Given a set P of N points, find $p,q \in P$, such that the distance d(p, q) is minimum
- Algorithms for determining the closest pair:
  - Brute Force $O(n^2)$
  - Divide and Conquer O(n log n)
  - Sweep-Line O(n log n)



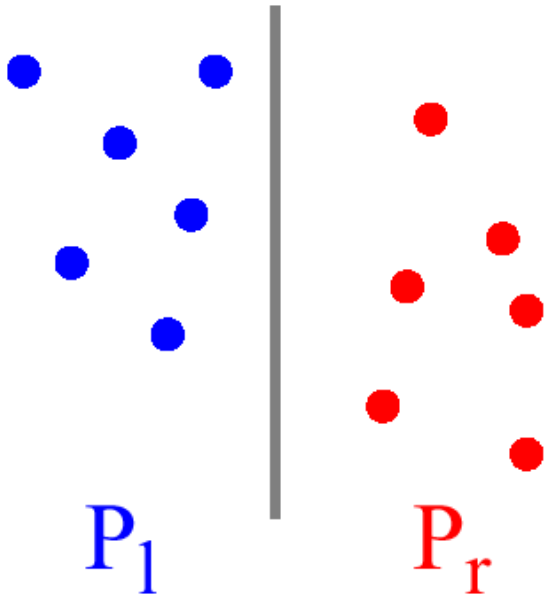
Brute Force

- Compute all the distances d(p,q) and select the minimum distance
- Running time **$O(n^2)$**



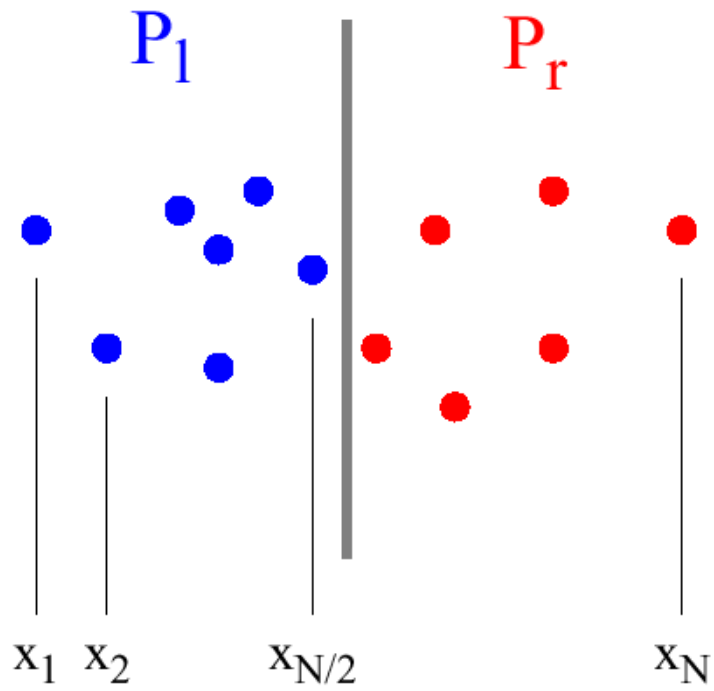
Divide and Conquer

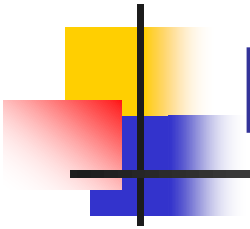
- **Sort points** on the x-coordinate and **divide them in half**
- Closest pair is either in one of the halves or has a member in each half



Divide and Conquer (2)

- **Phase** 1: Sort the points by their x-coordinate
- $p_1\ p_2 \cdots p_{n/2} \cdots p_{n/2+1} \cdots p_n$



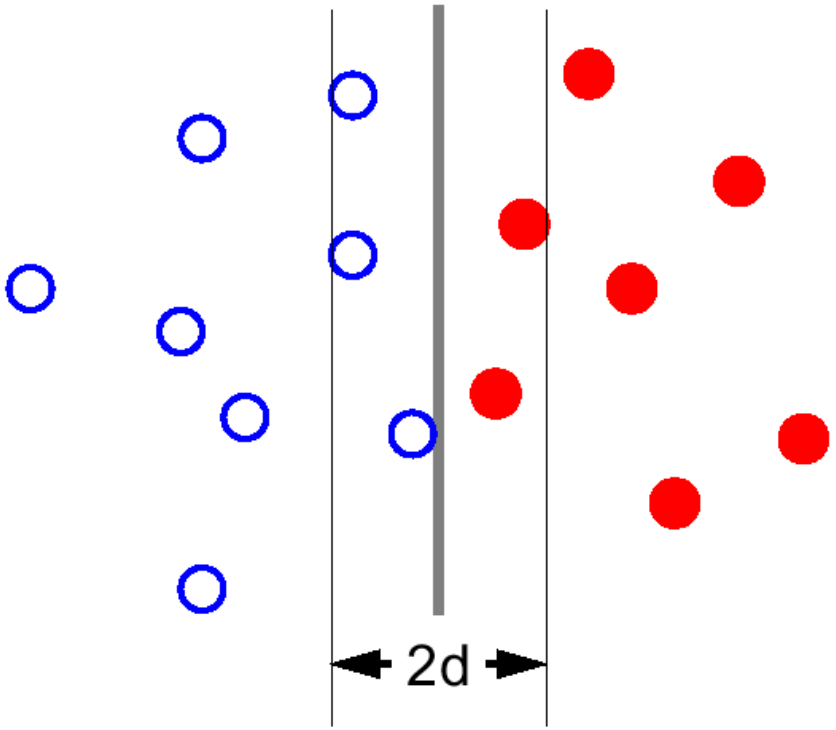


Divide and Conquer (3)

- **Phase 2:** Recursively compute closest pairs and minimum distances,
  - d_l , d_r in
  - $P_l = \{p_1, p_2, \ldots , p_{n/2} \}$
  - $P_r = \{p_{n/2+1}, \ldots , p_n \}$
- Find the closest pair and closest distance in central strip of width 2d, where
  - $d = \min(d_l, d_r)$
- in other words...

Divide and Conquer (4)

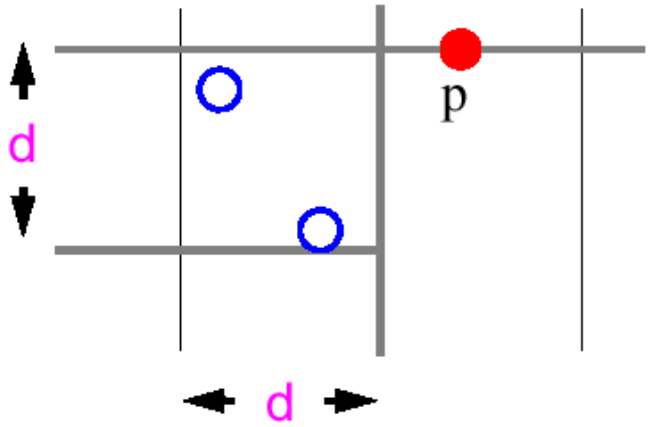
- Find the closest (○ , ●) pair in a strip of width **2d**, knowing that no (○ , ○) or (● , ●) pair is closer than d



Combining Two Solutions

- For each point p in the strip, check distances d(p, q), where p and q are of different colors and:

$$y(p) - d \le y(q) \le y(p)$$



- There are no more than four such points!

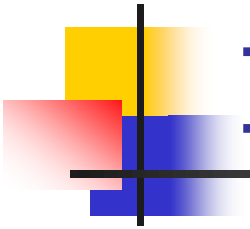
Running Time

- Sorting by the y-coordinate at each conquering step yields the following recurrence

$$T(n) = 2T(n/2) + n\log n$$
$$T(1) = 1$$

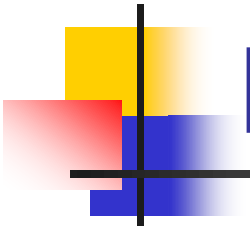
$O(n\log^2 n)$

$$\begin{aligned}
T(n) &= 2T(n/2) + n\log n \\
&= 4T(n/4) + 2(n/2)\log(n/2) + n\log n \\
&= 4T(n/4) + n(\log n - 1) + n\log n \\
&\ldots \\
&= 2^k T(n/2^k) + n(\log n + (\log n - 1) + \ldots + (\log n - k + 1)) \\
&\ldots \\
&\text{stop when } n/2^k = 1;\ k = \log n \\
&= n + n(1 + 2 + 3 + \ldots + \log n) \\
&= n + n((\log n + 1)\log n)/2
\end{aligned}$$



Improved Algorithm

- The idea: **Sort** all the points by y-coordinate **once**
- Before recursive calls, **partition the sorted list** into two sorted sublists for the left and right halves

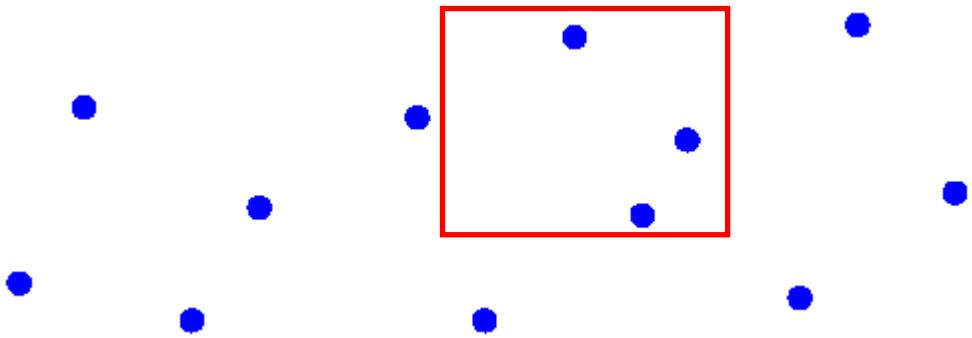


Running Time

- Phase 1:
  - Sort by x and y coordinate:
  - $O(n \log n)$
- Phase 2:
  - Partition: $O(n)$
  - Recur: $2T(n/2)$
  - Combine: $O(n)$
- $T(n) = 2\ T(n/2) + n = O(n \log n)$
- **Total Time: $O(n \log n)$**

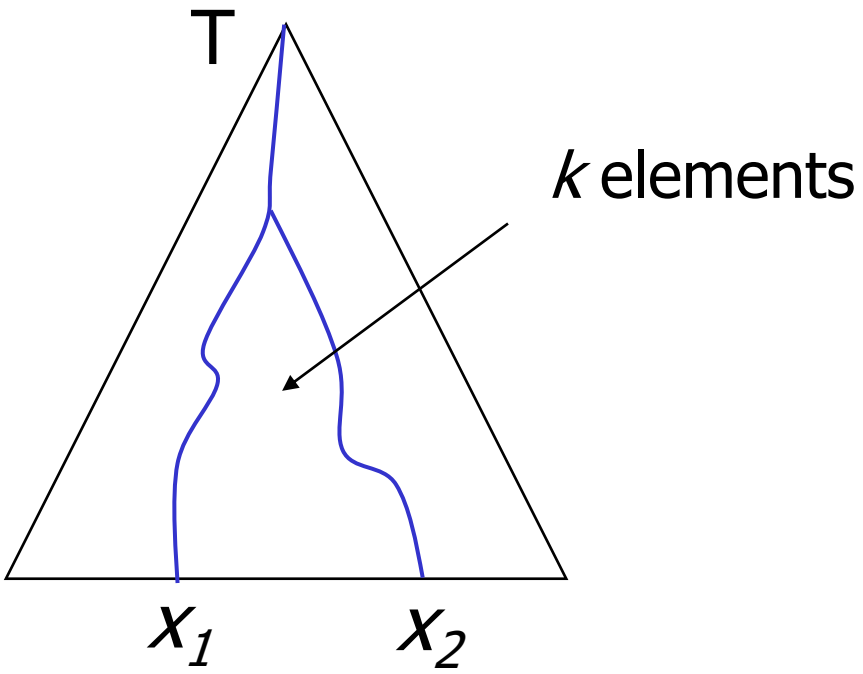
Multidimensional DS

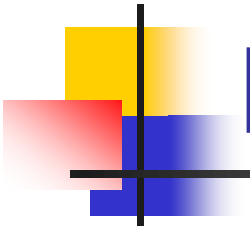
- Multidimensional Data Structures are used to answer queries on the set of multidimensional points
  - Range query ($[x_1, x_2]$, $[y_1, y_2]$): find all points (x, y) such that $x_1 < x < x_2$ and $y_1 < y < y_2$



1D Range Searching

- Find x, such that $x_1 < x < x_2$
  - Can be done with balanced binary search trees (e.g., red-black trees) in $O(\log n + k)$ time, where k is the size of the answer



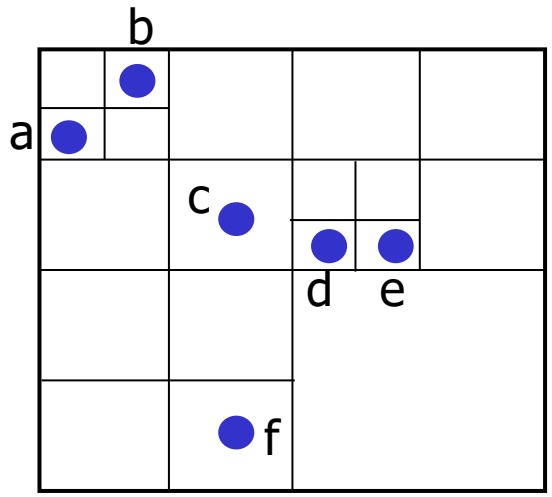


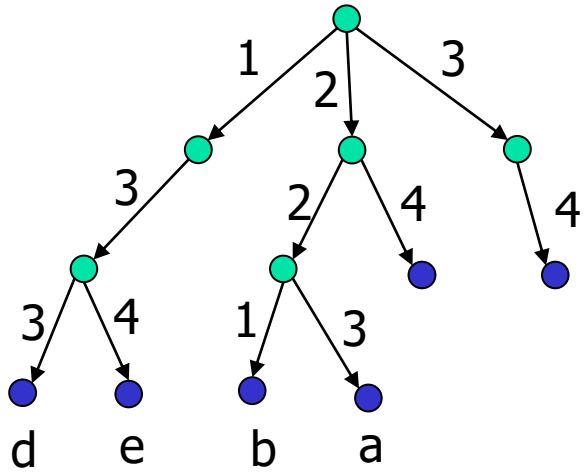
Range Trees

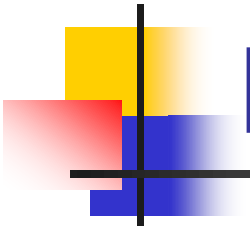
- Build a binary tree T on x-coordinates
- With each node v, associate a binary tree T_v that stores all the points in the subtree of T rooted at v. Organize T_v on y-coordinates.
- Space: $O(n \log n)$
- Range query: $O(\log^2 n + k)$

Quadtrees

- Quadtrees – partition tree
  - Linear space
  - Good average query performance







Next lecture

- Amortized Analysis